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A laminar jet in a rotating fluid

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A solution is presented of the asymptotic flow due to a point source of momentum in a uniformly rotating unbounded environment. Under the condition that the relative swirl velocity of the jet is small compared with the ambient swirl velocity, the equations of motion reduce to a set of linear equations. These equations are expressed in terms of similarity variables and a single ordinary differential equation is derived in terms of the similarity stream function. The profiles of the flow are calculated numerically.

The jet is shown to have a narrow viscous core whose thickness increases with distance z from the virtual source of momentum as $(\nu z/\Omega)^{\frac{1}{3}}$, where ν is the kinematic viscosity and Ω the ambient angular velocity.

Introduction

The effect of a strong ambient rotation on the behaviour of a developed laminar jet is considered. The ambient fluid is taken to be unbounded under the assumption that the presence of rigid boundaries rotating with the ambient fluid will not change the essential features of the flow. Although this assumption may restrict the relevance of the problem to the real physical situation, the results obtained below indicate that the restriction may not be too severe, especially for lateral boundaries. For it is found that the constraint imposed on the jet flow by the strong ambient rotation results in the radial mass flux decreasing to zero at large distances from the jet axis. Thus, the entire flow arising from the source of momentum is confined in the radial direction. It follows that lateral boundaries, provided that they are sufficiently distant from the jet axis, should be of little significance.

The influence of a transverse boundary behind the source of momentum is more difficult to assess, but provided that the source is placed in the interior of the fluid at a distance from the boundary, it should be negligibly small. It remains to consider what is probably the most important effect of all, namely, that due to a transverse boundary in front of the source. As the jet meets the boundary, fluid from the core attempts to spread radially. However, this motion is weak and takes place against the strong constraint on radial motion imposed by the ambient pressure field. Hence, the radial motion may be expected to be checked quickly and a weak reversed axial flow created outside the jet core. In such a situation, the influence of the transverse boundary is not significant throughout the entire flow but is limited to a region of the flow adjacent to the boundary. In

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this 'adjustment' region the influence of the boundary is decreased, by the action of viscosity, with increasing distance from the boundary. Provided that the distance between the source and the boundary is sufficiently great, there is a region of the flow beyond the 'adjustment' region in which it is a reasonable approximation to flow conditions to take the ambient fluid as unbounded.

Even though the results obtained may be of only limited application, they are of interest in themselves as they reveal several unexpected consequences of the strong rotational stratification.

The equations of motion

The behaviour of a round laminar jet of incompressible viscous fluid is considered. The ambient fluid is taken to be unbounded and in uniform rotation about the jet axis with angular velocity Ω. A system of cylindrical polar coordinates (r, θ, z), with origin at the virtual source of the jet, is used within a frame of reference which rotates with an angular velocity $(0, 0, \Omega)$. The velocity components, relative to the rotating frame, are denoted by (u, v, w).

Assuming the flow to be steady and axi-symmetric, the equations of motion are

$$u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}-2\Omega v = -\frac{\partial p}{\partial r}+\nu\left(Lu+\frac{\partial^2 u}{\partial z^2}\right), \tag{1}$$

$$u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}+2\Omega u = \nu\left(Lv+\frac{\partial^2 v}{\partial z^2}\right), \tag{2}$$

$$u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z}+\nu\left(L_1 w+\frac{\partial^2 w}{\partial z^2}\right), \tag{3}$$

where
$$L_1 \equiv L+\frac{1}{r^2} \equiv \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}.$$

Hence, relative to a fixed frame of reference, $\rho(p+\frac{1}{2}\Omega^2 r^2)$ is the pressure, and $v+\Omega r$ the swirl velocity, at a radial distance r. ρ is the density and ν the kinematic viscosity of the fluid.

The set of equations is completed by the continuity equation

$$\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial w}{\partial z} = 0. \tag{4}$$

Equation (3) may be integrated over a transverse plane to give an integral expression for the constant axial flux of momentum,

$$M = \int_0^{2\pi}\int_0^{\infty}\left\{rw^2+rp+\nu\frac{\partial}{\partial r}(ru)\right\}dr\,d\theta. \tag{5}$$

In deriving (5), it has been assumed that

$$ruw \to 0, \quad r\,\partial w/\partial r \to 0 \quad \text{as} \quad r \to \infty.$$

In the following analysis, the orders of magnitude of radial and axial lengths will be denoted by δ and l, respectively, and of all other quantities by the corresponding capital letter.

Consider first the continuity equation. The terms are of order U/δ and W/l, respectively, and so

$$\frac{U}{W} = O\left(\frac{\delta}{l}\right). \tag{i}$$

Using (i), the orders of magnitude of the terms in equations (1)–(3), relative to the inertia terms are

$$1, \quad 1, \quad \left(\frac{R_V}{R_U}\right)^2, \quad \frac{1}{R_o}\left(\frac{R_V}{R_U}\right)^2, \quad \frac{P}{U^2}, \quad \frac{1}{R_U}, \quad \frac{1}{R_U}\left(\frac{\delta}{l}\right)^2, \tag{6}$$

$$1, \quad 1, \quad 1, \quad \frac{1}{R_o}, \quad \frac{1}{R_U}, \quad \frac{1}{R_U}\left(\frac{\delta}{l}\right)^2, \tag{7}$$

$$1, \quad 1, \quad \frac{P}{U^2}\left(\frac{\delta}{l}\right)^2, \quad \frac{1}{R_U}, \quad \frac{1}{R_U}\left(\frac{\delta}{l}\right)^2. \tag{8}$$

The terms in the integral expression (5) are of order

$$M, \quad l^2U^2, \quad \delta^2P, \quad \nu\delta U. \tag{9}$$

In the above, $R_U = U\delta\nu^{-1}$, $R_V = V\delta\nu^{-1}$ and $R_o = V(\Omega\delta)^{-1}$. Thus, R_U and R_V have the form of Reynolds numbers for the flow. The Rossby number for the flow reduces to the approximate form R_o if $U \ll V$. As this condition is found to hold later, R_o may be considered as the relevant Rossby number.

The discussion is now limited to those jet flows in which the relative swirl velocity is small compared with the ambient swirl velocity, that is

$$R_o = o(1). \tag{ii}$$

There is also the condition arising from the boundary layer nature of the flow,

$$\delta/l = o(1). \tag{iii}$$

Using conditions (ii) and (iii) it follows from (7) that

$$R_U = O(R_o), \tag{iv}$$

and so equation (2) represents primarily a balance between viscous and Coriolis forces. The principal terms in equation (3) are now seen to be the pressure and viscous-force terms. These give rise to the condition

$$\frac{P}{U^2} = O\left(\frac{1}{R_o}\left(\frac{l}{\delta}\right)^2\right). \tag{v}$$

From these conditions, equation (1) is found to reduce to a balance between the Coriolis acceleration and the radial pressure-gradient. The condition arising from this balance is

$$R_V/R_o = O(l/\delta). \tag{vi}$$

Consider now the terms in the integral expression (5). The orders of magnitude of these terms, (9), may be expressed conveniently in non-dimensional forms by means of ν. They become, using conditions (iv) and (v)

$$M/\nu^2, \quad \{R_o(l/\delta)\}^2, \quad R_o(l/\delta)^2, \quad R_o, \tag{10}$$

where $M^{\frac{1}{2}}\nu^{-1}$ is the only non-dimensional grouping of the external parameters of the flow and may be considered as the overall Reynolds number for the flow. Hence

$$M/\nu^2 = O\{R_o(l/\delta)^2\}, \tag{vii}$$

and the most important contribution to the integral is seen to come from the pressure term. That is, the convection and viscous diffusion of the axial momentum flux are small compared with the effect of the pressure. This is in marked contrast to jets in a still environment where convection is most effective in transporting momentum.

The conditions (vi), (vii), (iv), (i) and (v) imply, in order, that

$$\delta = O\{(\nu l/\Omega)^{\frac{1}{3}}\}, \quad V = O(M/\nu l), \quad U = O\{M/(\Omega\nu^2)^{\frac{1}{3}} l^{\frac{5}{3}}\},$$
$$W = O(M/\nu l), \quad P = O\{M(\Omega/\nu l)^{\frac{2}{3}}\}. \tag{viii}$$

Hence, those jet flows for which R_o is small have a narrow viscous core whose thickness increases with axial distance z from the virtual source as $(\nu z/\Omega)^{\frac{1}{3}}$. It appears also from (viii) that the axial and relative swirl velocities are of the same order. Thus, the basic assumption of the analysis, that R_o is small, is seen to imply slow jet motion.

In the above analysis, no conditions have been imposed on the external parameters of the flow. It is not possible, therefore, to derive a condition in terms of these parameters alone for which the above reduction is valid. However, using (viii), condition (vii) may be rearranged to give an inequality in terms of z, namely

$$z \gg (M/\nu^2)^{\frac{3}{4}}(\nu/\Omega)^{\frac{1}{2}}. \tag{ix}$$

It follows that the reduced equations are valid in that region of the flow where condition (ix) is satisfied.

The reduced equations

The reduced equations of motion are, then,

$$2\Omega v = \partial p/\partial r, \tag{11}$$

$$2\Omega u = \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right), \tag{12}$$

$$0 = -\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right), \tag{13}$$

and the axial momentum flux is given by the expression

$$M = 2\pi\int_0^\infty rp\,dr. \tag{14}$$

The full boundary conditions, which are not necessarily independent, are

$$u = 0, \quad v = 0, \quad \partial w/\partial r = 0, \quad \partial p/\partial r = 0,$$

w and p are finite on $r = 0$.

$$(u, v, w) \to 0, \quad p \to 0, \quad \partial/\partial r\,(u, v, w) \to 0, \quad \partial p/\partial r \to 0$$

as $r \to \infty$.

A streamfunction ψ may be introduced by means of the continuity equation so that

$$u = \frac{1}{r}\frac{\partial \psi}{\partial z} \quad \text{and} \quad w = -\frac{1}{r}\frac{\partial \psi}{\partial r}.$$

The results of the orders-of-magnitude analysis (viii) indicate that an asymptotic solution of the reduced equations exists. They imply that at large axial distances from the virtual source of momentum, the profiles of the flow will be similar at different transverse sections. Thus, by suitable substitutions, the equations (11)–(13) may be simplified to ordinary differential equations by expressing them in terms of similarity functions.

The forms of the substitutions are given by (viii), and are:

$$\begin{aligned} \psi(r,z) &= \frac{M}{2\pi\nu}\left(\frac{\nu}{\Omega}\right)^{\frac{2}{3}} z^{-\frac{1}{3}} f(x), \\ \Gamma(r,z) \equiv rv &= \frac{M}{2\pi\nu}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{3}} z^{-\frac{2}{3}} h(x), \\ p(r,z) &= \frac{M}{2\pi}\left(\frac{\Omega}{\nu}\right)^{\frac{2}{3}} z^{-\frac{4}{3}} m(x), \end{aligned} \tag{15}$$

where $x = \frac{1}{2}(\Omega/\nu)^{\frac{2}{3}} r^2/z^{\frac{2}{3}}$.

The circulation $\Gamma(r,z)$ is introduced at this stage because it makes the reduced equations even in r. This permits the use of x as a similarity variable (rather than the more usual form of $x^{\frac{1}{2}}$) and hence a simpler form is obtained for the derived equations.

The substitution of (15) into equations (11)–(13) gives

$$h = xm' \tag{16}$$

$$-f - 2xf' = 3xh'', \tag{17}$$

$$0 = m + xm' - 3f'' - 3xf'''. \tag{18}$$

The integral expression (14) becomes

$$1 = \int_0^\infty m(x)\,dx. \tag{19}$$

In equations (16)–(18), primes denote differentiation with respect to x.

The velocity components are now of the form

$$u = -\frac{M}{6\sqrt{2}\,\pi\nu}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{3}} z^{-\frac{5}{3}}\left(x^{-\frac{1}{2}}f + 2x^{\frac{1}{2}}f'\right),$$

$$v = \frac{M}{2\sqrt{2}\,\pi\nu} z^{-1}x^{-\frac{1}{2}}h, \quad w = -\frac{M}{2\pi\nu} z^{-1} f'.$$

In terms of the similarity functions, the full boundary conditions are

$$\begin{aligned} &x^{-\frac{1}{2}}f(x) \to 0, \quad x^{-\frac{1}{2}}h(x) \to 0, \quad f'(x) \text{ and } m(x) \text{ finite as } x \to 0; \\ &x^{-\frac{1}{2}}f(x) \to 0, \quad x^{\frac{1}{2}}f'(x) \to 0, \quad xf''(x) \to 0, \\ &x^{-\frac{1}{2}}h(x) \to 0, \quad h'(x) \to 0, \quad m(x) \to 0, \\ &x^{\frac{1}{2}}m'(x) \to 0, \quad \text{as} \quad x \to \infty. \end{aligned}$$

Hence, from equation (18), it follows directly that $m(0) = 3f''(0)$ and on integration that

$$xm = 3xf'' + c,$$

where c is a constant which can be seen to be zero from the conditions on $x = 0$. Therefore,

$$m(x) = 3f''(x). \tag{20}$$

Substituting for $m(x)$ in equation (16) and (19) gives

$$h(x) = 3xf'''(x), \tag{21}$$

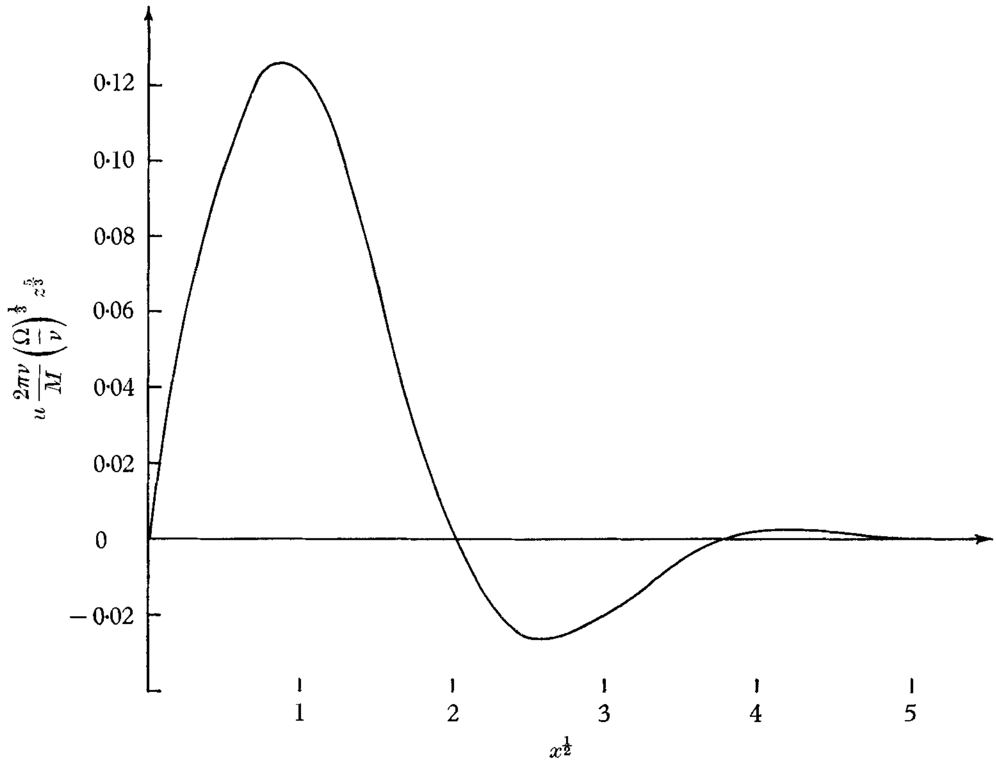

FIGURE 1. Radial velocity profile.

and $f'(0) = -\frac{1}{3}$. Finally, substituting for $h(x)$ in equation (17) gives a differential equation in the similarity stream-function alone

$$9x^2f^{\text{v}} + 18xf^{\text{iv}} + 2xf' + f = 0. \tag{22}$$

The boundary conditions on equation (22) are

$$x^{-\frac{1}{2}}f(x) \to 0 \quad \text{as} \quad x \to 0,$$
$$f'(0) = -\tfrac{1}{3}, \quad f''(0) \text{ is finite},$$
$$x^{-\frac{1}{2}}f(x) \to 0 \quad \text{and} \quad x^{\frac{1}{2}}f'(x) \to 0 \quad \text{as} \quad x \to \infty.$$

The solution of equation (22) was obtained by numerical integration. The details are given in the appendix. The velocity and pressure profiles are shown in figures 1, 2, 3 and 4, respectively, with an abscissa $x^{\frac{1}{2}}$ which is proportional to r.

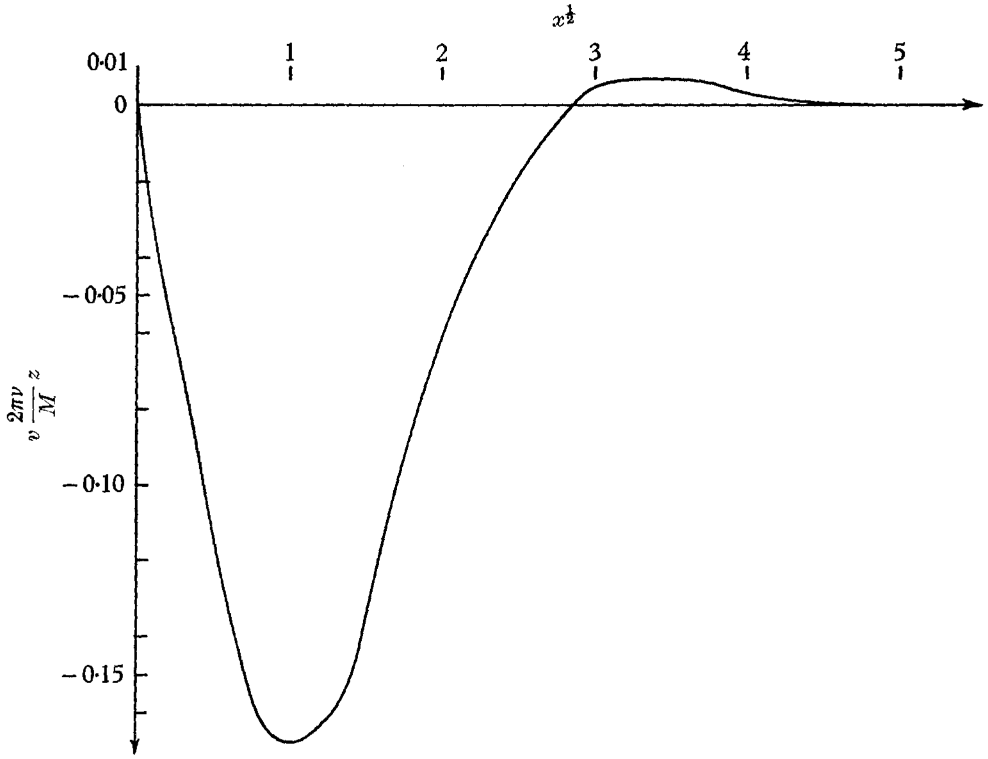

FIGURE 2. Swirl velocity profile.

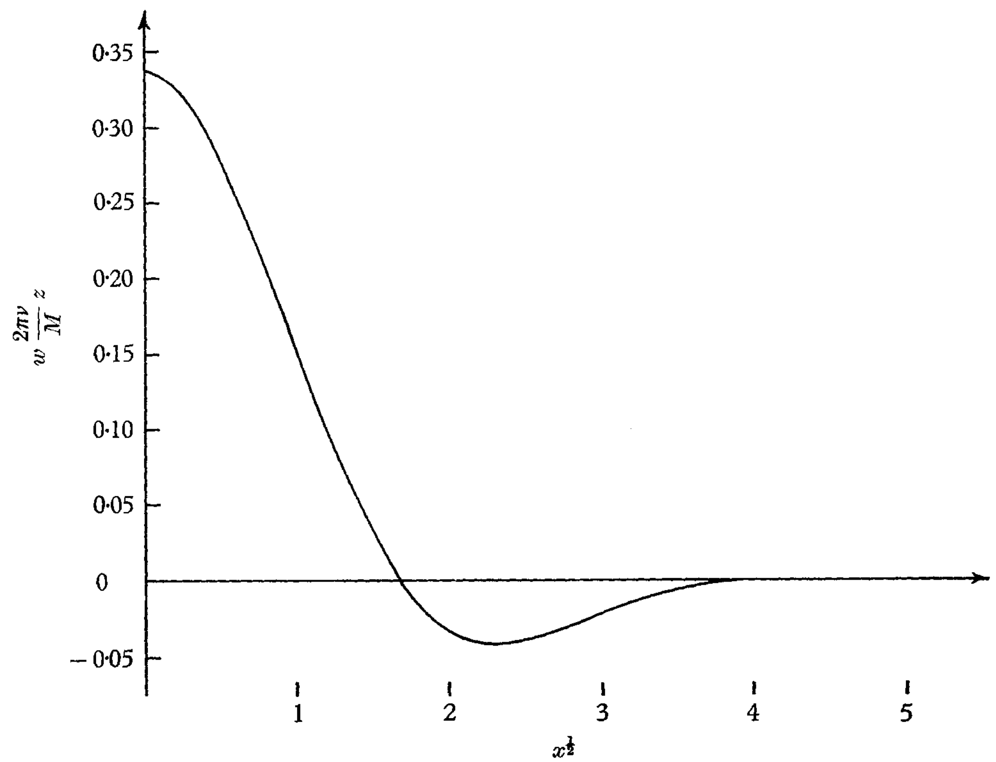

FIGURE 3. Axial velocity profile.

An important feature of the solution is that, although the condition $u \to 0$ was applied for large r, the much stronger condition $ru \to 0$ is found to hold. That is, the radial mass flux per unit axial length, ρru, decreases to zero as r increases. It follows that the entire flow arising from the source of momentum is confined in the radial direction. Using this result, integration of the continuity equation over a transverse plane leads to the condition

$$\rho \int_0^\infty rw\,dr = J,$$

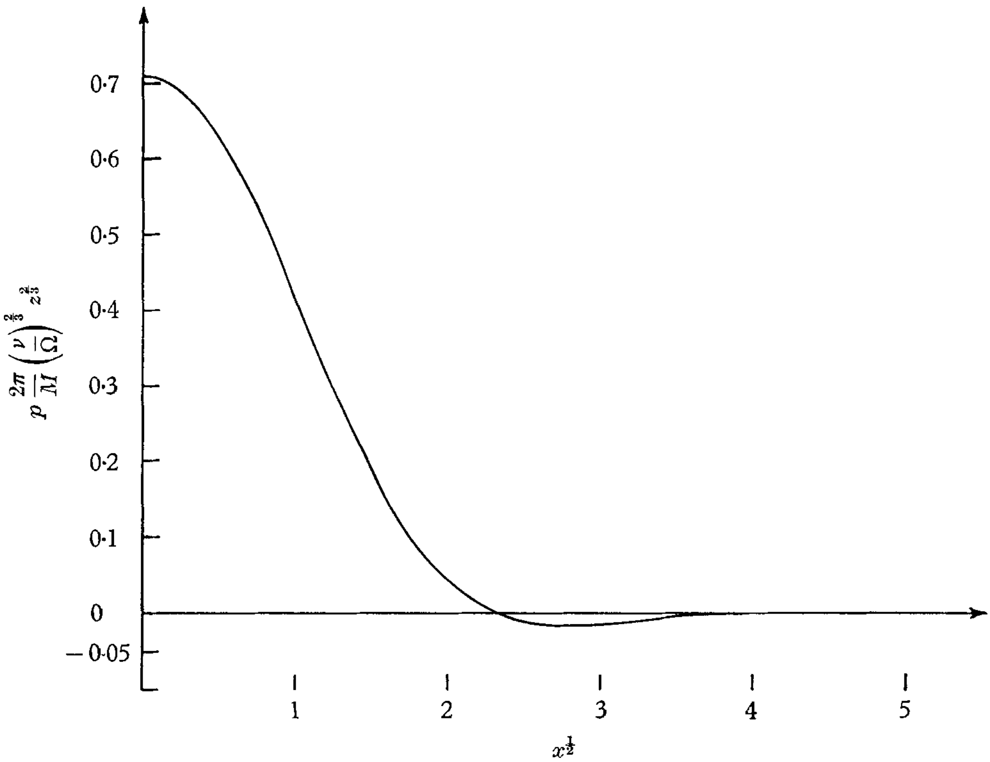

FIGURE 4. Pressure profile.

where J is the constant mass flux across transverse planes. However, when it is expressed in terms of the similarity variables, the integral is found to be dependent on z and so J must be zero. This result can, of course, be obtained directly from the series solutions (A 1), (A 2) of equation (22). It follows that there must be a region of reversed axial flow.

In the flow outside the core of the jet there is an approximate geostrophic balance and the radial flow is consequently weak. As this weak radial flow is towards the jet axis and as angular momentum is conserved, the swirl velocity in this region is increased slightly by the presence of the jet, i.e. the relative swirl velocity is positive but small. The relative radial pressure-gradient is therefore positive and small also. Since the relative pressure decreases to zero as r increases, it follows that the relative pressure outside the core is negative.

In the core of the jet there is a strong radial outflow (compared with the radial flow outside the jet) and so the relative swirl velocity is large and negative in this region of the flow. Hence, the relative pressure-gradient is also large and negative and this gives the relative pressure distribution a maximum on the jet axis.

It is of interest to compare these results with those obtained for a related problem by Long (1958, 1961). The flow considered by Long was that due to an intense vortex in an infinite viscous fluid. He assumed that the vorticity was concentrated into the core of the vortex, so that the circulation K outside the vortex core was independent of radius, and that Coriolis-force terms were negligibly small, i.e. $R_0 \gg 1$. Long found that the shape of the flow profiles changed radically as the non-dimensional parameter $N = M/k^2$ varied. He found also that, above a certain minimum value of N, the profiles were not even determined uniquely by N. This contrasts with the results given here in which the profiles are found to be similar for all flows of the type considered. Another point of difference is that, for the fast-swirling jet, the relative pressure was found to have a minimum value on the axis, as was to be expected from Bernoulli's equation. However, when the relative motion is small, a complete reversal is found with the relative pressure at a maximum on the axis.

Finally, there are two possible limiting processes to the flow. They are $M \to 0$ and $\Omega \to \infty$, the first of which is trivial. To obtain the second limit, a change of the independent variables r and z is required so that the similarity variable x does not involve Ω explicitly. This may be achieved by introducing the non-dimensional variables $\eta = r/a$ and $\zeta = z/a$ where $a = (\nu/\Omega)^{\frac{1}{2}}$. The similarity variable now takes the form $x = \eta^2/2\zeta^{\frac{2}{3}}$. When expressed in terms of η and ζ, it is found that the velocity components are proportional to $\Omega^{\frac{1}{2}}$ and the pressure to Ω. This is consistent with the result $\delta \propto \Omega^{-\frac{1}{3}}$ obtained above in that the core of the jet becomes narrower and the velocity components and relative pressure correspondingly larger as Ω is increased.

Appendix: Solution of the reduced equations

The expansion of $f(x)$ for small x is found by substituting the series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

into equation (22). This substitution leads to a recurrence relation,

$$\frac{a_{n+3}}{a_n} = \frac{-(2n+1)}{9(n+3)(n+2)(n+1)^2 n},$$

from which three independent solutions can be obtained. The other two solutions, involving $\log x$, do not satisfy the finiteness conditions on $x = 0$ and so are rejected.

The expansion of $f(x)$ which satisfies all the boundary conditions on $x = 0$ is

$$f(x) = -\tfrac{1}{3}x\left(1-\frac{x^3}{144}+\ldots\right) + a_2x^2\left(1-\frac{x^3}{648}+\ldots\right) + a_3x^3\left(1-\frac{7x^3}{12{,}960}+\ldots\right), \tag{A 1}$$

where a_2 and a_3 are constants which are determined by the required behaviour of the solution for large x.

The expansion of $f(x)$ for large x is found similarly. Substituting the series

$$f(x) = \sum_{n=0}^{\infty} b_n x^{-\alpha-n}$$

into equation (22) leads to the recurrence relation

$$\frac{b_{n+3}}{b_n} = -\frac{9(n+\frac{1}{2})(n+\frac{3}{2})(n+\frac{5}{2})^2(n+\frac{7}{2})}{2(n+3)},$$

with $\alpha = \frac{1}{2}$. This gives only one independent solution. The other four solutions may be found from a substitution of the form

$$f(x) = b_0 x^{-\frac{1}{2}} e^{cx^\beta},$$

where b_0, c and β are constants. This gives the full expansion of $f(x)$ for large x to be

$$f(x) = b_0 x^{-\frac{7}{4}}\left(1 - \frac{1575}{64x^3} + \ldots\right)$$
$$+ x^{-\frac{1}{2}} e^{-y}(b_1 \cos y + b_2 \sin y) + x^{-\frac{1}{2}} e^{y}(b_3 \cos y + b_4 \sin y), \qquad \text{(A 2)}$$

where $y = (2^7x^3/3^6)^{\frac{1}{4}}$ and the b_i $(i = 1, 2, 3, 4)$ are constants. The boundary conditions on $f(x)$ require that, as $x \to \infty$, the exponentially large terms be absent. It will now be shown, by considering the asymptotic expansion of (A 1), that a_2 and a_3 can be chosen to make b_3 and b_4 zero.

The expansion of $f(x)$ for small x can be expressed in terms of generalized hypergeometric functions, thus:

$$f(x) = -\tfrac{4}{27}\pi^{\frac{3}{2}} x \sum_{n=0}^{\infty} \frac{(n-\frac{1}{2})!}{(n+\frac{1}{3})!\{(n-\frac{1}{3})!\}^2(n-\frac{2}{3})!}\frac{\varpi_1^n}{n!}$$
$$+ \frac{2^{\frac{5}{3}}}{2^{\frac{3}{2}}}\pi^{\frac{1}{2}}(\tfrac{1}{3})!\,a_2x^2 \sum_{n=0}^{\infty} \frac{(n-\frac{1}{6})!}{(n+\frac{2}{3})!(n+\frac{1}{3})!(n-\frac{1}{3})!\,n!}\frac{\varpi_1^n}{n!}$$
$$+ \frac{2^{\frac{7}{3}}}{3^{\frac{5}{2}}}\pi^{\frac{1}{2}}(-\tfrac{1}{3})!\,a_3x^3 \sum_{n=0}^{\infty} \frac{(n+\frac{1}{6})!}{(n+1)!(n+\frac{2}{3})!\{(n+\frac{1}{3})!\}^2}\frac{\varpi_1^n}{n!}, \qquad \text{(A 3)}$$

where $\varpi_1 = -2x^3/3^6$.

The asymptotic expansions of these functions can be obtained from the general theory given by Wright (1935). It is found that, for large x

$$f(x) \sim (\alpha_1 - \alpha_2 a_2 \sin(\pi/12) - \alpha_3 a_3 \cos(\pi/12))\, x^{\frac{1}{4}} \cos y\,.\,e^y$$
$$+ (\alpha_1 + \alpha_2 a_2 \cos(\pi/12)) + \alpha_3 a_3 \sin(\pi/12)\, x^{\frac{1}{4}} \sin y\,.\,e^y,$$

where $\alpha_1 = -(2^7/3^6)^{\frac{1}{4}}, \quad \alpha_2 = 2^{\frac{19}{12}}3^2(\tfrac{1}{3})!\,\pi^{-1}$

and $\alpha_3 = 2^{\frac{23}{12}}3^3(-\tfrac{1}{3})!\,\pi^{-1}.$

The exponentially large terms vanish, therefore, when

$$a_2 = 2^{-\frac{1}{3}}3^{-2}(-\tfrac{1}{3})! \quad \text{and} \quad a_3 = -2^{-\frac{2}{3}}3^{-3}(\tfrac{1}{3})!.$$

The substitution of these values in (A 3) leads to an expression for $f(x)$ in terms of a single generalized hypergeometric function

$$f(x) = -\frac{2}{9}\sqrt{\frac{\pi}{3}}\, x \sum_{m=0}^{\infty} \frac{(\frac{1}{3}m-\frac{1}{2})!}{(\frac{1}{3}m-\frac{1}{3})!(\frac{1}{3}m+\frac{1}{3})!}\frac{\varpi_2^m}{m!}, \qquad \text{(A 4)}$$

where $\varpi_2 = -2^{\frac{1}{3}}x/3$.

However, the flow profiles were not obtained from (A4) directly because the series converges only slowly. Instead, as all the boundary conditions for small x were then known numerically, the profiles were obtained by numerical integration of equation (22) outwards from a small non-zero value of x (to avoid the singularity in the equation at $x = 0$).

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